Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2018

Calculus

Higher level

Paper 3

15 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685\ldots$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a - b)$	Do not award the final ***A1***

3 ***N*** marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets** eg **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (eg TI-89) are not allowed.

Calculator notation The mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) **METHOD 1**

$\ln(n+2) < n+2$ ***(A1)***

$\Rightarrow \dfrac{1}{\ln(n+2)} > \dfrac{1}{n+2}$ (for $n \geq 0$) ***A1***

Note: Award ***A0*** for statements such as $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)} > \sum_{n=0}^{\infty} \frac{1}{n+2}$. However condone such a statement if the above ***A1*** has already been awarded.

$\sum_{n=0}^{\infty} \dfrac{1}{n+2}$ (is a harmonic series which) diverges ***R1***

Note: The ***R1*** is independent of the ***A1***s. Award ***R0*** for statements such as "$\frac{1}{n+2}$ diverges".

so $\sum_{n=0}^{\infty} \dfrac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

METHOD 2

$\dfrac{1}{\ln n} > \dfrac{1}{n}$ (for $n \geq 2$) ***A1***

Note: Award ***A0*** for statements such as $\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$. However condone such a statement if the above ***A1*** has already been awarded.

a correct statement linking n and $n+2$ *eg*,

$\sum_{n=0}^{\infty} \dfrac{1}{\ln(n+2)} = \sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ or $\sum_{n=0}^{\infty} \dfrac{1}{n+2} = \sum_{n=2}^{\infty} \dfrac{1}{n}$ ***A1***

Note: Award ***A0*** for $\sum_{n=0}^{\infty} \frac{1}{n}$.

$\sum_{n=2}^{\infty} \dfrac{1}{n}$ (is a harmonic series which) diverges

(which implies that $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by the comparison test) ***R1***

Note: The ***R1*** is independent of the ***A1***s. Award ***R0*** for statements such as $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges and "$\frac{1}{n}$ diverges". Award ***A1A0R1*** for arguments based on $\sum_{n=1}^{\infty} \frac{1}{n}$.

so $\sum_{n=0}^{\infty} \dfrac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

continued…

Question 1 continued

(b) applying the ratio test $\lim\limits_{n\to\infty}\left|\frac{(3x)^{n+1}}{\ln(n+3)}\times\frac{\ln(n+2)}{(3x)^n}\right|$ ***M1***

$=|3x|$ (as $\lim\limits_{n\to\infty}\left|\frac{\ln(n+2)}{\ln(n+3)}\right|=1$) ***A1***

Note: Condone the absence of limits and modulus signs.

Note: Award ***M1A0*** for $3x^n$. Subsequent marks can be awarded.

series converges for $-\frac{1}{3}<x<\frac{1}{3}$

considering $x=-\frac{1}{3}$ and $x=\frac{1}{3}$ ***M1***

Note: Award ***M1*** to candidates who consider one endpoint.

when $x=\frac{1}{3}$, series is $\sum\limits_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ which is divergent (from (a)) ***A1***

Note: Award this ***A1*** if $\sum\limits_{n=0}^{\infty}\frac{1}{\ln(n+2)}$ is not stated but reference to part (a) is.

when $x=-\frac{1}{3}$, series is $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{\ln(n+2)}$ ***A1***

$\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{\ln(n+2)}$ converges (conditionally) by the alternating series test ***R1***

(strictly alternating, $|u_n|>|u_{n+1}|$ for $n\geq 0$ and $\lim\limits_{n\to\infty}(u_n)=0$)

so the interval of convergence of S is $-\frac{1}{3}\leq x<\frac{1}{3}$ ***A1***

Note: The final ***A1*** is dependent on previous ***A1***s – *ie*, considering correct series when $x=-\frac{1}{3}$ and $x=\frac{1}{3}$ and on the final ***R1***.

Award as above to candidates who firstly consider $x=-\frac{1}{3}$ and then state conditional convergence implies divergence at $x=\frac{1}{3}$.

[7 marks]

Total [10 marks]

2. considering continuity at $x=2$

$\lim\limits_{x\to 2^-} f(x)=1$ and $\lim\limits_{x\to 2^+} f(x)=4a+2b$ ***(M1)***

$4a+2b=1$ ***A1***

considering differentiability at $x=2$

$$f'(x)=\begin{cases}-1 & x<2\\ 2ax+b & x\geq 2\end{cases}$$ ***(M1)***

$\lim\limits_{x\to 2^-} f'(x)=-1$ and $\lim\limits_{x\to 2^+} f'(x)=4a+b$ ***(M1)***

> **Note:** The above ***M1*** is for attempting to find the left and right limit of their derived piecewise function at $x=2$.

$4a+b=-1$ ***A1***

$a=-\dfrac{3}{4}$ and $b=2$ ***A1***

[6 marks]

3. (a) $\displaystyle\int_4^\infty \frac{1}{x^3}\,\mathrm{d}x=\lim_{R\to\infty}\int_4^R \frac{1}{x^3}\,\mathrm{d}x$ ***(A1)***

> **Note:** The above ***A1*** for using a limit can be awarded at any stage.
> Condone the use of $\lim\limits_{x\to\infty}$.
> Do not award this mark to candidates who use ∞ as the upper limit throughout.

$$=\lim_{R\to\infty}\left[-\frac{1}{2}x^{-2}\right]_4^R\left(=\left[-\frac{1}{2}x^{-2}\right]_4^\infty\right)$$ ***M1***

$$=\lim_{R\to\infty}\left(-\frac{1}{2}\left(R^{-2}-4^{-2}\right)\right)$$

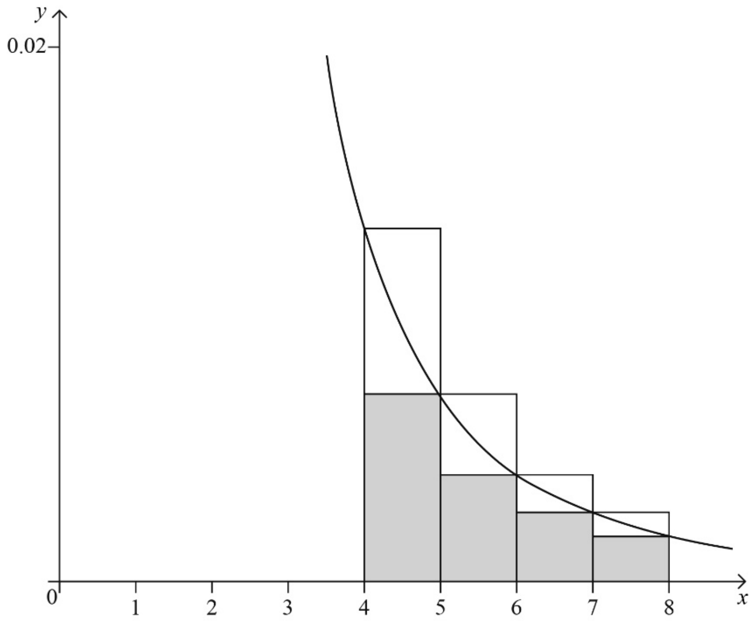
$$=\frac{1}{32}$$ ***A1***

[3 marks]

continued…

Question 3 continued

(b)



A1A1A1A1

A1 for the curve
A1 for rectangles starting at $x = 4$
A1 for at least three upper rectangles
A1 for at least three lower rectangles

Note: Award ***A0A1*** for two upper rectangles and two lower rectangles.

sum of areas of the lower rectangles $<$ the area under the curve $<$ the sum of the areas of the upper rectangles so

$$\sum_{n=5}^{\infty}\frac{1}{n^3} < \int_4^{\infty}\frac{1}{x^3}\,\mathrm{d}x < \sum_{n=4}^{\infty}\frac{1}{n^3}$$ ***AG***

[4 marks]

(c) a lower bound is $\frac{1}{32}$ ***A1***

Note: Allow ***FT*** from part (a).

[1 mark]

(d) **METHOD 1**

$$\sum_{n=5}^{\infty}\frac{1}{n^3} < \frac{1}{32}$$ ***(M1)***

$$\frac{1}{64} + \sum_{n=5}^{\infty}\frac{1}{n^3} < \frac{1}{32} + \frac{1}{64}$$ ***(M1)***

$\sum_{n=4}^{\infty}\frac{1}{n^3} < \frac{3}{64}$, an upper bound ***A1***

Note: Allow ***FT*** from part (a).

continued…

Question 3 continued

METHOD 2

changing the lower limit in the inequality in part (b) gives

$$\sum_{n=4}^{\infty}\frac{1}{n^3}<\int_3^{\infty}\frac{1}{x^3}\,\mathrm{d}x\left(<\sum_{n=3}^{\infty}\frac{1}{n^3}\right)$$ ***(A1)***

$$\sum_{n=4}^{\infty}\frac{1}{n^3}<\lim_{R\to\infty}\left[-\frac{1}{2}x^{-2}\right]_3^R$$ ***(M1)***

$\sum_{n=4}^{\infty}\frac{1}{n^3}<\frac{1}{18}$, an upper bound ***A1***

Note: Condone candidates who do not use a limit.

[3 marks]

Total [11 marks]

4. (a) $f'(x)=\dfrac{2\arcsin(x)}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for an attempt at chain rule differentiation.
Award ***M0A0*** for $f'(x)=2\arcsin(x)$.

$f'(0)=0$ ***AG***

[2 marks]

(b) differentiating gives $(1-x^2)f^{(3)}(x)-2xf''(x)-f'(x)-xf''(x)(=0)$ ***M1A1***

differentiating again gives $(1-x^2)f^{(4)}(x)-2xf^{(3)}(x)-3f''(x)-3xf^{(3)}(x)-f''(x)(=0)$

M1A1

Note: Award ***M1*** for an attempt at product rule differentiation of at least one product in each of the above two lines.
Do not penalise candidates who use poor notation.

$(1-x^2)f^{(4)}(x)-5xf^{(3)}(x)-4f''(x)=0$ ***AG***

[4 marks]

continued…

Question 4 continued

(c) attempting to find **one of** $f''(0)$, $f^{(3)}(0)$ or $f^{(4)}(0)$ by substituting $x=0$ into relevant differential equation(s) ***(M1)***

Note: Condone $f''(0)$ found by calculating $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\arcsin(x)}{\sqrt{1-x^2}}\right)$ at $x=0$.

$\left(f(0)=0, f'(0)=0\right)$

$f''(0)=2$ and $f^{(4)}(0)-4f''(0)=0 \Rightarrow f^{(4)}(0)=8$ ***A1***

$f^{(3)}(0)=0$ and so $\frac{2}{2!}x^2+\frac{8}{4!}x^4$ ***A1***

Note: Only award the above ***A1***, for correct first differentiation in part (b) leading to $f^{(3)}(0)=0$ stated or $f^{(3)}(0)=0$ seen from use of the general Maclaurin series. **Special Case:** Award ***(M1)A0A1*** if $f^{(4)}(0)=8$ is stated without justification or found by working backwards from the general Maclaurin series.

so the Maclaurin series for $f(x)$ up to and including the term in x^4 is $x^2+\frac{1}{3}x^4$

AG

[3 marks]

(d) substituting $x=\frac{1}{2}$ into $x^2+\frac{1}{3}x^4$ ***M1***

the series approximation gives a value of $\frac{13}{48}$

so $\pi^2 \simeq \frac{13}{48}\times 36$

$\simeq 9.75\left(\simeq \frac{39}{4}\right)$ ***A1***

Note: Accept 9.76.

[2 marks]

Total [11 marks]

5. (a) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x^{p-1} + \frac{1}{x}$ ***(M1)***

integrating factor $= \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-\ln x}$ ***(A1)***

$= \frac{1}{x}$ ***A1***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = x^{p-2} + \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = x^{p-2} + \frac{1}{x^2}$

$\frac{y}{x} = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of C.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1$, $y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = \frac{1}{p-1}\left(x^p - x\right) - 1$ ***A1***

[8 marks]

continued…

Question 5 continued

METHOD 2

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1(A1)***

substituting, ***M1***

$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - vx = x^p + 1$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-1} + \frac{1}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-2} + \frac{1}{x^2}$

$v = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of C.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1$, $y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = \frac{1}{p-1}\left(x^p - x\right) - 1$ ***A1***

[8 marks]

(b) (i) **METHOD 1**

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p-1}\left(px^{p-1} - 1\right)$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow px^{p-1} - 1 = 0$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

continued…

Question 5 continued

METHOD 2

substitute $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and their y into the differential equation and solve for x

$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow -\left(\frac{x^p-x}{p-1}\right)+1=x^p+1$ ***M1***

$x^p-x=x^p-px^p$ ***A1***

$px^{p-1}=1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1}=\frac{1}{p}$ ***AG***

(ii) there are two solutions for x when p is odd (and $p>1$) ***A1***

if $p-1$ is even there are two solutions (to $x^{p-1}=\frac{1}{p}$)

and if $p-1$ is odd there is only one solution (to $x^{p-1}=\frac{1}{p}$) ***R1***

Note: Only award the ***R1*** if both cases are considered.

[4 marks]

Total [12 marks]